Example 4.2:

Find a MacLaurin series for the following function:

$$g(x) = \frac{4x}{2+x}$$

(Where does it converge?)

$\rightarrow$ look at the series for this function:

$$\frac{1}{2+x} = \frac{1}{2(1+\frac{x}{2})} = \frac{1}{2} \cdot \underline{\sum_{n=0}^{\infty} (-1)^n \frac{x^n}{2^n}} \text{ for } \left|\frac{x}{2}\right| < 1$$

geometric

$r = -\frac{x}{2}$

$\rightarrow$ so $g(x) = \dfrac{4x}{2+x}$

$$= 2 \cdot \sum_{N=0}^{\infty} \frac{(-1)^N X^{N+1}}{2^N}$$

$$\frac{1}{1-r} = \sum_{N=0}^{\infty} r^N$$

$$= \sum_{N=0}^{\infty} (-1)^N \frac{X^{N+1}}{2^{N-1}}$$

$$= \sum_{N=1}^{\infty} (-1)^{N-1} \frac{X^N}{2^{N-2}}$$

Example: Find a MacLaurin series for $f(x) = \cos(2x)$

1. $2\sum_{k}(-1)^{k}\frac{x^{2k}}{(2k)!}$
2. $\sum_{k}(-1)^{k}\frac{x^{2k+2}}{k!}$
3. $\sum_{k}(-1)^{k}\frac{2^{k}x^{2k}}{(2k)!}$
4. $\boxed{\sum_{k}(-1)^{k}\frac{4^{k}x^{2k}}{(2k)!}}$

- for all real t

$$\cos(t) = \sum_{N=0}^{\infty}\frac{(-1)^N t^{2N}}{(2N)!}$$

- take $t = 2x$:

$$f(x) = \cos(2x) = \sum_{N=0}^{\infty}\frac{(-1)^N 2^{2N} x^{2N}}{(2N)!}$$

$$(2^2)^N = 4^N$$

$$= \sum_{N=0}^{\infty}\frac{(-1)^N 4^N x^{2N}}{(2N)!}$$

Recall: Differentiation and Integration of Power Series

$$\frac{d}{dx}\left(\sum_{k=0}^{\infty} a_k x^k\right) = \sum_{k=1}^{\infty} k a_k x^{k-1} = \sum_{k=0}^{\infty} a_k \frac{d}{dx}\left[x^k\right]$$

$$\int\left(\sum_{k=0}^{\infty} a_k x^k\right)dx = \sum_{k=0}^{\infty} \frac{a_k}{k+1} x^{k+1} + C = \sum_{k=0}^{\infty} a_k \int_0^x t^k\,dt$$

The radius and interval of convergence are preserved under differentiation and Integration.

Example 5:

Find a MacLaurin Series for the function $f(x) = x^2 \tan^{-1}(x)$

$\rightarrow$ find a MacLaurin series for $\tan^{-1}(x)$:

$$\tan^{-1}(x) = \int_0^x \frac{dt}{1+t^2} \leftarrow$$

geometric series

$r = -t^2$

$|r| < 1 \Longleftrightarrow |t| < 1$

$\Longleftrightarrow |x| < 1$

$$= \int_0^x \left(\sum_{n=0}^{\infty} (-1)^n t^{2n} \right) dt$$

$$= \sum_{N=0}^{\infty} (-1)^N \int_0^x t^{2N}\,dt$$

$$= \sum_{N=0}^{\infty} (-1)^N \left. \frac{t^{2N+1}}{2N+1} \right|_0^x = \sum_{N=0}^{\infty} (-1)^N \frac{x^{2N+1}}{(2N+1)}$$

$\longrightarrow$ so $f(x) = x^2 \cdot \tan^{-1}(x)$

$$= x^2 \sum_{N=0}^{\infty} (-1)^N \frac{x^{2N+1}}{2N+1} = \sum_{N=0}^{\infty} \frac{(-1)^N x^{2N+3}}{2N+1}$$

Example: Find a power series for $\int_0^x \cos(t^2)\,dt = f(x)$

1. $\sum_k (-1)^k \frac{x^{4k+1}}{(4k+1)(2k)!}$
2. $\sum_k (-1)^k \frac{x^{4k+4}}{(4k+4)(2k)!}$
3. $\sum_k (-1)^k \frac{x^{4k^2+1}}{(4k^2+1)(2k)!}$
4. $\sum_k (-1)^k \frac{x^{2k+1}}{(2k+1)(2k)!}$

$\rightarrow$ first find a power series for $\cos(t^2)$

$$\cos(t^2) = \sum_{N=0}^{\infty} \frac{(-1)^N t^{4N}}{(2N)!} \qquad (t^2)^{2N} = t^{4N}$$

$\rightarrow$ integrate termwise:

$$\int_0^x \cos(t^2)\,dt$$

$$= \sum_{N=0}^{\infty} \frac{(-1)^N}{(2N)!} \int_0^x t^{4N}\,dt = \sum_{N=0}^{\infty} \frac{(-1)^N}{(2N)!} \frac{t^{4N+1}}{4N+1} \Big|_0^x$$

$$= \sum_{N=0}^{\infty} \frac{(-1)^N}{(2N)!} \frac{x^{4N+1}}{4N+1}$$

Example 6:

Estimate $I = \int_0^{1/2} \cos(x^3)\,dx$ within an error range of 0.001.

$\rightarrow$ idea: find a series for the integral and approximate it to within $\frac{1}{1000}$ using methods we have seen

$$\to \cos(x^3) = \sum_{N=0}^{\infty} \frac{(-1)^N x^{6N}}{(2N)!}$$

$$\to I = \int_0^{1/2} \cos(x^3)\,dx = \sum_{N=0}^{\infty} \frac{(-1)^N}{(2N)!} \int_0^{1/2} x^{6N}\,dx$$

$$= \sum_{N=0}^{\infty} \frac{(-1)^N}{(2N)!} \left. \frac{x^{6N+1}}{6N+1} \right|_0^{1/2}$$

$$= \sum_{N=0}^{\infty} \frac{(-1)^N}{(2N)!} \cdot \frac{1}{2^{6N+1}} \cdot \frac{1}{6N+1} \quad (*)$$

→ apply an approx. to the alternating series in (*).

$$\left| I - \sum_{k=0}^{N} \frac{(-1)^k}{(2k)!} \cdot \frac{1}{2^{6k+1}} \cdot \frac{1}{6k+1} \right|$$

$$\leq \frac{1}{(2N+2)!} \cdot \frac{1}{2^{6N+7}} \cdot \frac{1}{6N+7} \leq \frac{1}{1000}$$

Plug in $N = 0, 1, 2$ to find N so this is true

N=2 works, so our approximation to I is:

$$I \approx \sum_{k=0}^{2} \frac{(-1)^k}{(2k)!} \cdot \frac{1}{2^{6k+1}} \cdot \frac{1}{6k+1}$$

6. (a) (4 points) Find a Taylor series centered at $a = 0$ (MacLaurin series) for the function $f(x) = e^{-\frac{x^2}{2}}$. Simplify your answer and write in $\sum$-notation.

(b) (4 points) Using your answer from part (a), find a series representation for the integral below. Simplify your answer to receive full credit.

$$\int_0^1 e^{-\frac{x^2}{2}}\,dx$$

(c) (4 points) Find the approximate value of your series in part (b) by estimating the sum within an error range of $E = 0.1$.

(a) recall that for any real t

$$e^t = \sum_{N=0}^{\infty} \frac{t^N}{N!}.$$

Plug in $t = -\frac{x^2}{2}$:

$$f(x) = e^{-\frac{x^2}{2}} = \sum_{N=0}^{\infty} \frac{(-1)^N x^{2N}}{2^N \cdot N!} \quad (*)$$

(b) • find a series for $I = \int_0^1 e^{-x^2/2}\,dx.$

• integrate $(*)$ termwise

$$I = \int_0^1 f(x)\,dx = \sum_{N=0}^{\infty} \frac{(-1)^N}{2^N \cdot N!} \int_0^1 x^{2N}\,dx$$

$$= \sum_{N=0}^{\infty} \frac{(-1)^N}{2^N \cdot N!} \left. \frac{x^{2N+1}}{2N+1} \right|_0^1$$

$$= \sum_{N=0}^{\infty} \frac{(-1)^N}{2^N \cdot N!} \cdot \frac{1}{(2N+1)} \quad (**)$$

(c) asked to approximate $(**)$ to within an error of $E = \frac{1}{10}$.

- by the alternating series approximation

$$\left| I - \sum_{k=0}^{N} \frac{(-1)^k}{2^k \cdot k!} \cdot \frac{1}{(2k+1)} \right| \leq \frac{1}{2^{N+1}(N+1)!} \cdot \frac{1}{(2N+3)}$$

$$\leq \frac{1}{10} \quad (***)$$

- plug in $N=0,1,2,\dots$ to find the smallest N so that $(***)$ is true:

$\rightarrow N=0: \quad \frac{1}{2\cdot 1} \cdot \frac{1}{3} = \frac{1}{6} \not\leq \frac{1}{10}$ ✗

$\rightarrow N=1: \quad \frac{1}{4\cdot 2} \cdot \frac{1}{5} = \frac{1}{40} \leq \frac{1}{10}$ ✓

- So we get that:

$$I \simeq \sum_{k=0}^{1} \frac{(-1)^k}{2^k \cdot k!} \cdot \frac{1}{(2k+1)}$$